6/19/89

WILLIAMSTOWN CALIBRATIONS CONFERENCE

August 15–17, 1988

Open Problems List

General Reference:

[PSPM] Proceedings of Symposia in Pure Mathematics, Vol. 44. *Geometric Measure Theory and the Calculus of Variations.* "Some Open Problems in Geometric Measure Theory and it Applications...", J. E. Brothers, ed. pp. 441 – 464.

1. **Finding big faces of the Grassmannian of k–planes in $\mathbb{R}^n$.**

We may as well let $k \leq n-k$. Then the largest possible faces have dimension at most $(n-k)(k-1)$. By looking at the second fundamental form on the Grassmannian, it can be seen that if a face of that dimension exists, then there exists a set of k–1 orthogonal skew–symmetric matrices whose pairwise products are skew–symmetric. These must be (n–k) by (n–k). Nicholas Katz at Princeton tells me that these exist for any k.* The size n–k may have to be large compared to k. It is not yet known whether we will get forms with large faces from this. Katz's method uses Clifford algebras. [G. Lawlor]

* It has been shown that there exist m–tableaux having this structure and hence, potentially, faces, iff n–k is the dimension of a Clifford module (R. Bryant, unpublished notes). Moreover, all these tableaux are involutive (see # 15). [J. M. Landsberg]

See also [PSPM], # 6.6.

2. **Estimate sizes of largest faces.**

What are the largest faces of $G_o(k, \mathbb{R}^n)$? The answer is known for $n \leq 8$. In $G_o(2, \mathbb{R}^{2n})$ or $G_o(2, \mathbb{R}^{2n+1})$ the largest face is a $\mathbb{C}P^n$ of complex lines, which maximize a Kahler form $e^*_{12} + e^*_{34} + \ldots + e^*_{2n-1,2n}$. In $G_o(3, \mathbb{R}^6)$ the largest face is that of the special Lagrangian form $e^*_{123} - e^*_{156} - e^*_{426} - e^*_{453}$. In $G_o(3, \mathbb{R}^7)$ [and $G_o(4, \mathbb{R}^7)$, by duality], the largest face belongs to an "associative form," without loss of generality

$$\varphi_A = e^*_{123} - e^*_{156} - e^*_{147} - e^*_{257} - e^*_{367} - e^*_{426} - e^*_{453}$$

and has dimension 8. Finally, in $G_o(4, \mathbb{R}^8)$ the largest face belongs to the "Cayley form",

$$\varphi_C = e^*_{1234} + e^*_{1256} + e^*_{1357} + e^*_{1458} + e^*_{2367} + e^*_{2468} + e^*_{3478}$$
$$- e^*_{1278} - e^*_{1638} - e^*_{1674} - e^*_{5238} - e^*_{5274} - e^*_{5634} + e^*_{5678}$$

and has dimension 12. Both of the last two examples are related to octonions or Cayley numbers; the formula for the first is

$$\langle x \wedge y \wedge z, \varphi_A \rangle = x \cdot yz,$$

where x, y, and z are imaginary octonions, and the second is

$$\langle x \wedge y \wedge z \wedge w, \varphi_C \rangle = \mathrm{Re}\,(\bar{x}y)(\bar{z}w),$$

provided x, y, z and w are orthonormal octonions. For $n \geq 9$ and $3 \leq k \leq n-3$, the largest face of $G_o(k, \mathbb{R}^n)$ is unknown. It is trivial to prove, using the first–cousin principle, that codim $(G_\varphi) \geq \max\{k, n-k\}$ for any calibration $\varphi$. I have a proof that for all $\varphi \in \Lambda^k\mathbb{R}^{2k}$, codim $(G_\varphi) \geq 2k-4$ provided that all pairs of planes in $G_\varphi$ satisfy an "angle equality" $\theta_1 \pm \theta_2 \pm \ldots \pm \theta_k = 0$. In addition, if k is even then the largest face has codimension at most $\frac{1}{2}(k^2-k+2)$, the codimension of the face of the special Lagrangian form in $\Lambda^2\mathbb{R}^{2k}$.

Questions suggested by the above information are:

Does the codimension of the largest face in $\Lambda^k\mathbb{R}^{2k}$ grow linearly in k (as suggested by the associative and Cayley forms), quadratically (as suggested by the special Lagrangian form), or somewhere in between? Do all pairs of planes in "sufficiently large" faces satisfy an angle equality? Do all "sufficiently large" faces have interesting transitive group actions on them? Is it true that if $G_\varphi$ is "sufficiently large," then there is a basis for $\Lambda^k\mathbb{R}^n$ in which every coordinate k–plane $\zeta$ satisfies $\langle \zeta, \varphi \rangle = -1$, 0 or 1? If any of these questions have "yes" answers, then how large is "sufficiently large"?

One of the difficulties with this problem is the fact that a simple dimension count imposes very little structure on $G_\varphi$. I believe that any real progress on the first question will require a positive answer to at least one of the remaining questions. (D. Mackenzie)

3. General methods for finding comass.

At present the only result resembling a "general method" is the Torus Lemma. In addition, formulas are known for comass in $\Lambda^2\mathbb{R}^n$, $\Lambda^3\mathbb{R}^6$ and $\Lambda^3\mathbb{R}^7$ (in the latter cases, the formulas are actually in the form of equivalent conditions for comass 1). In a few particular cases the norm–preserving properties of quaternion and octonion multiplication have been an aid to showing the comass of a form is 1. It is probably hopeless to expect to find a formula for comass in $\Lambda^4\mathbb{R}^8$. However, one can hope for methods that would simplify the calculation of comass in special cases, say when $\varphi$ satisfies some symmetry conditions or has a large number of zero components. This would seem to be essential for making progress on Problems 2 and 14. (D. Mackenzie)

4. How to find coflat calibrations.

Subsidiary questions:

How to find simple calibrations (at each point the form is dual to a single k–plane)

How to find nonsimple, nonconstant–coefficient calibrations

How to calibrate unoriented surfaces

References:

Federer, H. Real flat chains, cochains and variational problems. Indiana Univ. Math J. **24** (1974), 351–407, esp. 405–407.

Cheng, Benny. Ph. D. Thesis, M.I.T., 1987. Area–Minimizing Equivariant Cones and Coflat Calibrations

Murdoch, Timothy. Twisted Calibrations and the Cone on the Veronese Surface. Ph. D. Thesis, Rice University, 1988.

Lawlor, Gary. A Sufficient Criterion for a Cone to be Area–Minimizing. Ph. D. Thesis, Stanford University, 1988.

Description:

Federer defined a class of generalized differential forms allowing certain singularities, called coflat forms. He used a coflat form to reproduce Bombieri, De Giorgi and Giusti's result that the cone over $S^3 \times S^3$ is area–minimizing.

More broadly, we are interested in finding any calibrations that "work" in spite of their singularities. One such class is the set of exterior derivatives of Lipschitz forms. ([L] (see reference list above), appendix, Thm. A8 and A9.) These are not all coflat forms, because they may not be continuous at $\mathcal{H}^{n-1}$–almost all points of $\mathbb{R}^n$.

Benny's thesis discusses coflat calibrations and constructs examples. He exhibits nonsimple, nonconstant–coefficient calibrations, as does Tim Murdoch in his thesis. These are more powerful in general than simple calibrations, such as are used in my thesis; my methods failed for the cone on the Veronese surface, for example. On the other hand, they are more difficult to find (in my opinion). Can a localized construction be found which depends only on such things as curvature and normal radius (see [L] chapter 1) to construct nonsimple calibrations? This might yield such results as:

"Which triples of planes are area–minimizing?"

"Which pairs of planes are area–minimizing without regard to orientation?"

Tim Murdoch introduces a new type of calibration which allows him to deal with unorientable surfaces. He proves, as an example, that the cone on the Veronese surface ($\mathbb{R}P^2$) is minimizing among a fairly general class of comparison surfaces.

In my thesis, I construct area–nonincreasing retractions onto cones. For each of these there is a singular calibration which is dual at each point to the k–plane orthogonal to the (n–k)–dimensional surface through that point which retracts to a single point on the cone. The calibration can be (and originally was) constructed independently of the retraction. My favorite method is (1) start with a form $\omega$ which is closed and which we want to modify. (2) Find a good representative $\psi$ such that $d\psi = \omega$. (3) Modify $\psi$; for

example, we could multiply it by a function g. (4) Let $\varphi = d(g\psi)$ or d(modified $\psi$). This is automatically closed. If the modified $\psi$ is Lipschitz and the singularities of $\varphi$ don't intersect your surface ($\mathcal{H}^k$ almost everywhere) then you are in business; you only have to check comass. This may be the hard part if the resulting form $\varphi$ is not simple. [G. Lawlor]

5. **Find a calibration without $\Sigma\, \varphi_i^2 = 1$ (cf. Harvey–Lawson).**

6. **Methods for finding $\varphi$–manifolds.**

7. **If M is minimal, is $M^p$ *mimimizing* for some p?**

Related question: If B is a regular (or singular) minimal current in the sphere, is $0 \neq (B \times B \times \ldots \times B)$ minimizing if we take the Cartesian product enough times?

The answer is yes for the related question when B is regular.

The answer is no, sometimes, if M is not compact. Example: Let M be a pair of parallel lines with opposite orientation.

The answer if M is compact is a definite maybe. I believe it is true, at least if M is regular. One method is to try to find a (k–1)–parameter family of paths covering the k–dimensional surface M, intersecting only (possibly) at their starting points, and diverging (in cross–sectional (k–1)–volume) from each other. Using these, one would define a set of paths on $M^p$, and using the paths, one would construct an area–decreasing retraction from $\mathbb{R}^{np}$ to $M^p$. [G. Lawlor]

8. **Calibrations in semi–Riemannian geometries.**

I. A. Which pairs of "future–pointing", (n, 0)–planes in $\mathbb{R}^{n,n}$ are maximizing?

B. Which pairs of "future–pointing", (n, 0)–planes in $\mathbb{R}^{n,n}$ can be calibrated?

For example, if $\xi_0, \xi_1 \in$ SLAG, then $(\xi_0, \xi_1) \in$ B. Of course, B $\subset$ A, but they may not be equal.

II. $\phi_k \equiv -\frac{1}{2}\omega_I^2 - \frac{1}{2}\omega_J^2 + \frac{1}{2}\omega_K^2$ on $H^{1,q}$, where the $\omega$'s are the semi–Kahler forms on $\mathbb{C}^{2,2q} \cong H^{1,q}$ associated with $I = R_i$ (right multiplication), etc. Is $\phi_k$ a calibration?

Compare: $\phi \equiv$ .... (*Kahler* forms) *is* a calibration on $H^n$. (Bryant–Harvey)

III. Describe the cone C of (up to scale) calibrations on $\mathbb{R}^{p,q}$. (i.e. $C \subset \Lambda^p\mathbb{R}^{p,q}$)

Compare: *Any* (non–zero) k–form on $\mathbb{R}^n$ is a calibration (up to scale). [J. Mealy]

9. Regularity theory for calibrated surfaces.

10. Is the cross product of calibrations a calibration?

This is related to [PSPM], Problem 3.7: "Is the Cartesian product of two area minimizing surfaces area minimizing? For normal currents, a proof has been given in certain low dimensions and codimensions [M1, 1.3, 5.2]. However, for integral currents (or similarly for oriented manifolds with boundary), Almgren has given a counterexample (see [M2, Introduction]). The question is completely open for flat chains modulo 2. [F. Morgan]"

*References*:

[M1] F. Morgan, "The exterior algebra of $\Lambda^k\mathbb{R}^n$ and area minimization." Linear Algebra Appl. **66** (1985), 1–28.

[M2] F. Morgan, "Area–minimizing currents bounded by higher multiples of curves." Rend. Circ. Mat. Palermo (2) **33** (1984), 1–10.

11. **Area–minimizing triples of planes.**

The conjecture might be that an oriented triple of k–planes is area–minimizing if and only if each pair is area–minimizing. My current conjecture is that the triple may not be area–minimizing even if the pairs are. Frank found a triple of 3–planes in $\mathbb{R}^6$ which is pairwise special Lagrangian (for different SLAG forms) but not simultaneously special Lagrangian. This triple is not calibrated by a constant–coefficient form, but we do not know whether it is area–minimizing.

*References*: Lawlor, thesis, chapter 5.3.

Morgan, On the singular structure of three–dimensional area–minimizing surfaces. Transactions A. M. S. **276** (1983), pp. 137–143.

[G. Lawlor]

Perhaps a triple of 3–planes through the origin in $\mathbb{R}^6$ is never area–minimizing unless the three planes are simultaneously special Lagrangian? [D. Mackenzie]

F. Morgan points out that the above comment is a special case of the conjecture that a sum of planes is area–minimizing if and only if it is calibrated by a constant–coefficient calibration.

12. **Find a fractional–dimensional singularity of an area–minimizer, or show none exist.**

This question was suggested to me by two observations. (1) The most popular way to generate "fractals" these days is by iterating maps. (2) If you could find area–reducing maps with more complicated dynamics than Gary Lawlor's (which are projections, i.e. $F \circ F = F$) then one might get area–minimizers with fractal singularities. [D. Mackenzie]

From [PSPM]:

"Problem 5.4. Is it possible for the singular set of an area–minimizing integral (real) rectifiable current to be a Cantor type set with possibly non–integer Hausdorff

dimension? The area problem possibly has enough self–similarities to generate such a singular set. Cantor type singular sets of various dimensions between one and two can be realized for two–dimensional surfaces in $\mathbb{R}^3$ minimizing the integral of a convex (but not uniformly so) parametric integrand according to [TJ]. [F. Almgren]"

*Reference*:

[TJ] J. Taylor, Crystalline variational problems. Bull. A. M. S. 84 (1978), 568–588.

13. **Characterize cones calibrated by constant–coefficient calibrations.**

14. **Characteristic classes, especially $p_1$.**

Certain interesting parallel calibrations on Grassmann manifolds are given by the Pontriagin forms. The first interesting case is the Pontriagin form $p_1$ on $G_o(3, \mathbb{R}^6)$. Restricted to the tangent plane to $G_o(3, \mathbb{R}^6)$ at a point $\zeta$,

$$p_1 = e_{12}^* \wedge f_{12}^* + e_{12}^* \wedge g_{12}^* + f_{12}^* \wedge g_{12}^* + e_{13}^* \wedge f_{13}^* + e_{13}^* \wedge g_{13}^* + f_{13}^* \wedge g_{13}^* + e_{23}^* \wedge f_{23}^* + e_{23}^* \wedge g_{23}^* + f_{23}^* \wedge g_{23}^*,$$

where $\{e_1, e_2, e_3, f_1, f_2, f_3, g_1, g_2, g_3\}$ is an appropriately chosen basis. Gluck and/or Morgan conjectured that the comass of this form is $\sqrt{3/2}$, which is achieved at the simple 4–vector

$$\frac{1}{\sqrt{3}}(e_1 + f_2 + g_3) \wedge \frac{1}{\sqrt{2}}(e_2 - f_1) \wedge \frac{1}{\sqrt{2}}(e_3 - g_1) \wedge \frac{1}{\sqrt{2}}(f_3 - g_2).$$

Moreover, they have produced a singular surface in $G_o(3, \mathbb{R}^6)$ which is calibrated by this form, provided that its comass is indeed $\sqrt{3/2}$. [See #15 for more details.] Recently (March 1989) I have proved that this conjecture is correct. The open problems now are to understand this proof better, generalize to Pontriagin forms in higher–dimensional manifolds, and produce more examples of surfaces minimizing in their homology classes in this way. Will they also have singularities? [D. Mackenzie]

**15. New and used methods for proving minimality/area minimization.**

Here are some questions that I'd like to see answered with respect to the various methods that have been used to *find* volume–minimizing currents, and to *prove* that they are volume minimizing. They move from the more specific to the more general.

1. One way to find a volume–minimizer is to guess a candidate, and then calibrate your guess. Conversely, have a calibration and be skillful enough to find out what it calibrates. There are frequently natural candidates for volume–minimizers, for example, $G_{2l}\mathbb{R}^k \subset G_{2l}\mathbb{R}^n$ (for $l \geq 2$, $n \geq k > 2l$) [Gl–Mo–Zi], or the Hopf vector field on $S^3$ [Gl–Zi]. Life gets interesting when the natural candidates don't work out.

For example, $G_2\mathbb{R}^4$ does not minimize volume in its homology class in $G_3\mathbb{R}^6$. Does the cycle consisting of all geodesic segments from $e_1e_2e_3$ to $f_1f_2f_3$ in $G_3\mathbb{R}^6$ minimize volume in this homology class? Is this cycle calibrated by the Pontriagin form discussed by Dana Nance at Williams? (This is equivalent to proving that the comass of the form $\varphi$ Nance gave there is $\sqrt{3/2}$.)

The Hopf vector fields are not the minimum volume unit vector fields on $S^5$ [Jo]. There may not exist unit vector fields of minimum volume on $S^5$. Is the singular unit vector field $W^5$ described in [Pe] volume–minimizing in its homology class in $US^5$? (Same questions hold for $S^7$, $S^9$, etc.)

2. Tasaki [Ta] has shown that, in a compact, simple, simply–connected Lie group with bi–invariant metric, the cut locus to a point is calibrated by the dual of the fundamental 3–form, and hence is volume–minimizing in its homology class. Find more cases where the cut locus to a point, or currents contained in the cut locus to a point, is minimizing. (This makes finding cut loci explicitly and interesting problem. In particular, Gluck [Gl] has described the geodesics of $US^n$. Describe the cut locus to a point of $US^n$.)

3. Generalize Lawlor's [Lawl] curvature criterion to work on manifolds, not just Euclidean space. That is, take a subvariety (of some manifold other than Euclidean space)

which is the exponential image of its tangent cone at a singular point, and find a sufficient condition, involving curvature, second fundamental form, dimension, etc., for the subvariety to be minimizing. Note that, for example, $H^5$, $H^7$ and $\mathbb{H}^7$ (using the notation of [Pe] are all volume–minimizing cycles which are the exponential images of their tangent cones.

4. Given a manifold, Fomenko [Fo] provides a way to compute a lower bound (not necessarily sharp) for the volume of any representative of a k–dimensional homology class of the manifold. His method in some sense involves identifying the directions of maximum curvature in the manifold. He proves that, for example, $\mathbb{R}P^k$ is volume–minimizing in $\mathbb{R}P^n$, for all values of k and n. Push Fomenko's method and estimates beyond the symmetric cases that he works out.

5. Can Lawson's equivariant approach to the Plateau problem in Euclidean space [Laws] be modified to work in manifolds as well? There are two aspects of his method. One, use symmetries of the problem to reduce dimensions. Two, show that for a Plateau problem with equivariant data, the minimizer among all equivariant solutions is also the minimizer among all solutions of any type.

6. Can we get some other methods that don't boil down to calibrations? This is necessary for the classes we know where some multiple of the class is calibrated, but we know the class itself is not. I classify the foliation method of [Bo–deG–Gi], the equivariant method of [Laws], the oriented case of [Lawl] as boiling down to calibrations; [Fo] and the unoriented case of [Lawl] as not.

**Bibliography.**

[Bo–deG–Gi] E. Bombieri, E. de Giorgi and E. Giusti, "Minimal Cones and the Bernstein Problem," Inventiones Math. **7** (1969), 243–268.

[Fo] A. T. Fomenko, "Minimal Compacta in Riemannian Manifolds and the Reifenberg Conjecture," Mat. USSR Izvestija **6** (1972), 1037–66.

[Gl] Herman Gluck, "Geodesics in the Unit Tangent Bundle of a Round Sphere," to appear in L'Enseignement Math.

[Gl–Mo–Zi] Herman Gluck, Frank Morgan and Wolfgang Ziller, "Calibrated Geometries in Grassmann Manifolds," Comm. Math. Helv., to appear.

[Jo] David L. Johnson, "Volumes of Flows," to appear.

[Lawl] Gary Lawlor, *The Curvature Criterion*, Ph. D. Thesis, Stanford University (1988).

[Laws] H. Blaine Lawson, Jr., "The Equivariant Plateau Problem and Interior Regularity," Transactions of the A. M. S. **173** (1972), 231–249.

[Pe] Sharon L. Pedersen, *Optimal Vector Fields on Spheres*, Ph. D. Thesis, Univ. of Pennsylvania (1988).

[Ta] Hiroyuki Tasaki, "Certain minimal or homologically volume minimizing submanifolds in compact symmetric spaces," Tsukuba J. Math **9** (1985), 117–31.

[S. Pedersen]

*Also related to #15:*

In [L], a new technique for studying minimal submanifolds is used to construct examples of minimal submanifolds in $E^{2n+1}$. The submanifolds are characterized by having their Gauss map's image lie in degenerate SO(3) orbits of $G_{p,2n+1}$, the Grassmannian of p–planes in $E^{2n+1}$ (where the action on $G_{p,2n+1}$ is induced from the irreducible SO(3) action on $\mathbb{R}^{2n+1}$). The submanifolds are all given explicitly in terms of holomorphic data and are linearly full in $E^{2n+1}$. The degenerate SO(3) orbits are similar to, but not faces. They are examples of something we will call *m–subsets.*

Definition: $\Sigma \subset G_{p,n}$ is said to be an *m–subset* if

i. For all maps $f: M \to E^n$ such that the induced Gauss map $\gamma_f: M \to G_{p,n}$ has $\gamma_f(M) \subset \Sigma$, f(M) is a minimal submanifold of $E^n$.

Furthermore, $\Sigma$ is said to be an *involutive m–subset* if

ii. No proper submanifold of $\Sigma$ has the same size local parameter space of solutions as $\Sigma$.

i.e. Analytic solution submanifolds M such that $\gamma(M) \subset \Sigma$ will depend locally on an $s_l$ parameter family of functions of l variables and we require that no proper submanifold of $\Sigma$ have the same local solutions.

Faces are subsets of $G_{p,n}$ determined by calibrations, satisfying:

i'. For all maps $f: M \to E^n$ such that $\gamma_f(M) \subset \Sigma$, f(M) is an area–minimizing submanifold of $E^n$.

Call a face *involutive* if in addition it satisfies (ii). Call any subset (not necessarily determined by a calibration) satisfying (i') an *M–subset* and those satisfying (ii) as well an *involutive M–subset.*

If one is concerned with finding minimal or minimizing submanifolds there is no loss of generality in restricting attention to involutive m– or M–subsets because the solution submanifolds to any m– or M–subset, $\Sigma$, will have their Gauss map's image lying in some involutive subset of $\Sigma$.

Some questions:

1. Can one prove the fundamental lemma of calibrations up on the Grassmannian (i.e. only referring to the face and not its solution submanifolds or its calibration)? This would help in determining if there are any M–subsets that are not faces and which m–subsets are M–subsets.

2. Solution submanifolds to m–subsets are often given in terms of arbitrary functions. Given an m– but not M–subset whose local solutions are given by functions $f_j$, can we say which solutions will be area–minimizing by putting growth bounds on the $f_j$? This appears possible for the solutions of [L] using new methods of Gary Lawlor.

3. The tangent spaces to involutive m–subsets are examples of vector subspaces called *involutive m–tableaux*. One might hope to classify involutive m–tableaux. This would place strong restrictions on what m–subsets (and therefore nonsingular faces) could occur. For example, in [L] it is shown that there are no three dimensional involutive m–subsets (and therefore no three dimensional involutive faces) in **any** $G_{p,n}$. Progress has been made in the classification, but a complete list is still far off. For example, the case of 6–dimensional involutive m–tableaux in $G_{3,7}$ is still not understood. From preliminary results one might conjecture that the only irreducible involutive m–tableaux are those coming from faces.

*Reference*

[L] Landsberg, J. M. Minimal Submanifolds of $E^{2n+1}$ Arising From Degenerate SO(3) Orbits on the Grassmannian. Submitted to Transactions of the A. M. S.

[J. M. Landsberg]

16. **Pairs of unoriented planes.**

Which pairs of m–planes are area–minimizing in the class of unoriented surfaces? A necessary condition, conjectured to be sufficient, is that all orientations are area–minimizing in the class of oriented surfaces. [F. Morgan]

17. **Which calibrations qualify mod $\nu$?**

18. **$S^1 \otimes S^2$, $S^1 \otimes S^1 \otimes S^1$, Veronese cone.**

These are cones for which the criterion of my thesis failed. The first is the set of rank 1, 2 × 3 matrices. the second is described by the set of points in $\mathbb{R}^8$ of the form

$$a(c_1c_2c_3,\ c_1c_2s_3,\ c_1s_2c_3,\ c_1s_2s_3,\ s_1c_2c_3,\ s_1c_2s_3,\ s_1s_2c_3,\ s_1s_2s_3)$$

where $c_1^2 + s_i^2 = 1$ and $a \geq 0$.

The Veronese cone is the first example of Tim Murdoch's twisted–calibrated geometry. See reference in question #4. [G. Lawlor]

19. **3–dimensional soap films in $\mathbb{R}^4$ (singularities).**

From [PSPM]:

"Problem 5.14. **Classify (M, $\epsilon$, $\delta$)–minimal cones in higher dimensions.** Two–dimensional (M, $\epsilon$, $\delta$)–minimal cones in $\mathbb{R}^3$ (or a $C^{1,\alpha}$ three–dimensional manifold) have been classified. To my knowledge, there are no higher dimensional results. A reasonable conjecture is that the cone over the (k–2)–dimensional skeleton of a standard k–dimensional simplex is (M, $\epsilon$, $\delta$) minimizing in $\mathbb{R}^k$. Is it true? If so, then products of these with lines and planes are also minimizing; does this exhaust the list of possibilities up to dimension seven? [J. Taylor]"

*References*:

J. E. Taylor, The structure of singularities in soap–bubble–like and soap–film–like minimal surfaces. Ann. Math. (2) **103** (1976), 489–539.

———, The structure of singularities in solutions to ellipsoidal variational problems with constraints in $\mathbb{R}^3$, Ann. Math. (2) **103** (1976), 541–546.

20. **Existence of area–minimizing soap films.**

*Reference*: F. Morgan, "Soap films and mathematics," preprint.

21. **Methods for proving size–minimization.**

*Reference*: F. Morgan, "Size–minimizing rectifiable currents," Inventiones Math., to appear.

22. **Foliations about cone over $S(U_1 \times U_1 \times \ldots \times U_1)$.**

23. **Counterexample to Bernstein theorem for elliptic integrands.**

*Reference*: F. Morgan, "The cone over the Clifford torus in $\mathbb{R}^4$ is $\phi$–minimizing," preprint.

24. **Bernstein problem in affine geometry.**

25. **Douglas–Rado problem for $\phi$–minimal disks.**

From PSPM, Vol. 44:

"Problem 4.7. What regularity holds for the two–dimensional mapping problem (as in the classical Plateau problem) in which the area integrand is replaced by a parametric integrand? The existence of a minimizer with square integrable first derivatives was established by Morrey and others; see [MC. Chapter 9] for a discussion. [R. Hardt]"

*Reference*:

[MC] C. B. Morrey. *Multiple Integrals in the Calculus of Variations*, Springer–Verlag, 1966.

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