Egyptian Fractions

By Nam Nguyen ‘19

Numbers and basic computation appeared in Ancient Egypt as early as 2700 BCE. But you might not know that Ancient Egyptians demanded that every fraction have 1 in the numerator. They wanted to write any rational between 0 and 1 as a sum of such “unit” fractions. Such sums are called *Egyptian fractions*.

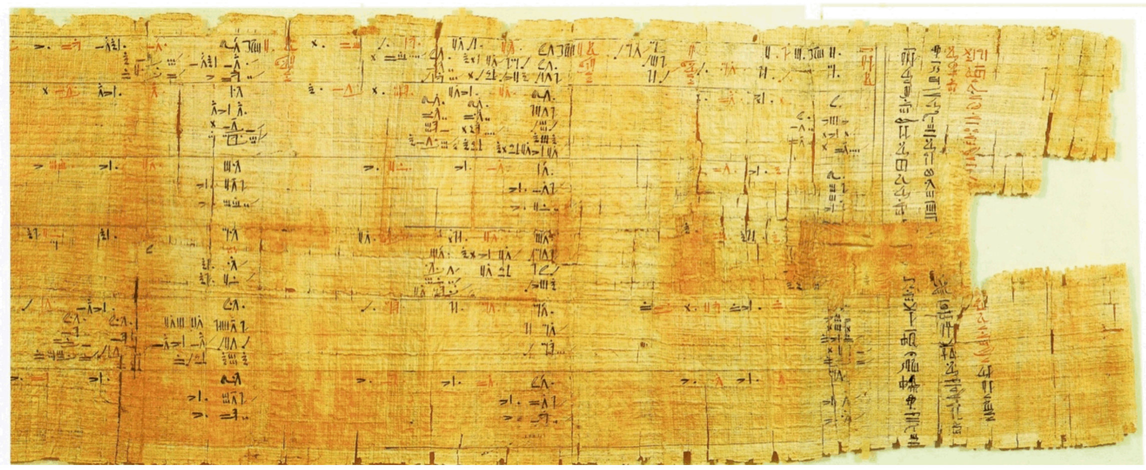

Figure 1: Egyptian fractions written on a papyrus scroll [1]

There are many ways to write 2/3 as an Egyptian fraction:

$$\frac{2}{3}=\frac{1}{3}+\frac{1}{3}$$

or

$$\frac{2}{3}=\frac{1}{3}+\frac{1}{5}+\frac{1}{20}+\frac{1}{12}$$

or

$$\frac{2}{3}=\frac{1}{3}+\frac{1}{6}+\frac{1}{30}+\frac{1}{20}+\frac{1}{12}.$$

Can you express every rational between 0 and 1 as an Egyptian fraction? It turns out that you can. The proof lies with the following greedy algorithm introduced by Fibonacci in 1202 in his book *Liber Albaci*:

At every stage, write down the largest possible unit fraction that is smaller than the fraction you are working on. For example, if you start with 11/12, the largest possible unit fraction that is smaller is 1/2. So

$$\frac{11}{12}=\frac{1}{2}+\frac{5}{12}.$$

The largest possible unit fraction that is smaller than 5/12 is 1/3. So

$$\frac{11}{12} = \frac{1}{2} + \frac{1}{3} + \frac{1}{12}.$$

The algorithm ends here because 11/12 is already expressed as a finite series of unit fractions.

More generally, given any fraction p/q, apply the Greedy algorithm to obtain

$$\frac{p}{q} - \frac{1}{u_1} = \frac{pu_1 - q}{qu_1},$$

where $1/u_1$ is the largest unit fraction below p/q. For convenience, we call $(pu_1 - q)/qu_1$ the remainder. Since

$$\lim_{u_1 \to \infty} 1/u_1 = 0 \text{ and } p/q > 0,$$

we know that u_1 always exists. Since $1/u_1$ is the largest unit fraction below p/q,

$$\frac{1}{u_1 - 1} > \frac{p}{q},$$

which is equivalent to $p > pu_1 - q$. Thus the numerator $pu_1 - q$ of the remainder is smaller than the original numerator p. If the remainder has 1 as the numerator, the process is finished. Otherwise, repeat the algorithm on the remainder. Since p is a positive integer, this process must inevitably terminate with a numerator of 1.

Another interesting goal is an Egyptian fraction of fewest terms. Figure 2 presents the smallest numbers of unit fractions required to express a rational p/q with $2 \le p \le 29$ and $3 \le q \le 30$.

\q: p\	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
2	2	-	2	-	2	-	2	-	2	-	2	-	2	-	2	-	2	-	2	-	2	-	2	-	2	-	2	-
3	.	2	2	-	3	2	-	2	2	-	3	2	-	2	2	-	3	2	-	2	2	-	2	2	-	2	2	-
4	.	.	3	-	2	-	2	-	2	-	3	-	2	-	3	-	2	-	2	-	2	-	3	-	2	-	3	-
5	.	.	.	2	3	2	2	-	3	2	3	2	-	2	3	2	2	-	2	3	3	2	-	2	2	2	2	-
6	.	.	.	.	3	-	-	-	2	-	3	-	-	-	2	-	3	-	-	-	2	-	2	-	-	-	2	-
7	.	.	.	.	.	3	3	2	3	2	2	-	3	3	3	2	3	2	-	3	3	2	3	2	2	-	3	2
8	.	.	.	.	.	.	3	-	4	-	3	-	2	-	4	-	3	-	2	-	2	-	3	-	3	-	3	-
9	.	.	.	.	.	.	.	3	4	-	3	2	-	2	2	-	4	2	-	3	3	-	3	2	-	2	3	-
10	.	.	.	.	.	.	.	.	4	-	3	-	-	-	3	-	2	-	2	-	3	-	-	-	2	-	2	-
11	.	.	.	.	.	.	.	.	.	3	3	3	3	3	3	2	3	2	2	-	3	2	3	3	3	2	3	2
12	.	.	.	.	.	.	.	.	.	.	4	-	-	-	3	-	3	-	-	-	2	-	4	-	-	-	4	-
13	.	.	.	.	.	.	.	.	.	.	.	4	3	3	3	3	3	3	3	2	3	2	2	-	3	3	3	2
14	.	.	.	.	.	.	.	.	.	.	.	.	3	-	4	-	4	-	-	-	3	-	3	-	2	-	4	-
15	.	.	.	.	.	.	.	.	.	.	.	.	.	4	4	-	4	-	-	3	4	-	-	2	-	2	2	-
16	.	.	.	.	.	.	.	.	.	.	.	.	.	.	5	-	3	-	3	-	4	-	3	-	3	-	3	-
17	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	3	4	3	3	3	4	3	3	3	3	3	3	2
18	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	4	-	-	-	4	-	3	-	-	-	4	-
19	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	3	3	3	4	3	3	4	3	3	4	3
20	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	4	-	4	-	-	-	4	-	4	-
21	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	4	5	-	3	4	-	-	4	-
22	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	5	-	4	-	4	-	3	-
23	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	3	4	4	3	3	4	3
24	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	4	-	-	-	4	-
25	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	4	4	3	4	-
26	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	4	-	4	-
27	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	4	5	-
28	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	5	-
29	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	4

Figure 2: Shortest Egyptian fraction lengths [1]

Though the concept of Egyptian fraction seems quite trivial at first, there are a few problems related to Egyptian fractions that have not been solved yet. A well-known example is the Erdös-Straus conjecture which states that:

Every rational of the form 4/n can be written as a sum of three unit fractions.

The conjecture has been verified for all values of n up to 10^{14}, yet no one has proved it true for *all values of n* or found a number n for which it is *not* true. The most recent progress toward proving this conjecture was made by Terrence Tao and Christian Elsholtz [2]. In their joint paper "Counting the number of solutions to the Erdös-Straus equation on unit fractions", they conclude that the number $f(n)$ of ways to write $4/n$ as a sum of three unit fractions satisfies the inequalities

$$N \log^2 N \ll \sum_{p \leq N} f(p) \ll N \log^2 N \log\log N,$$

where N is a sufficiently large positive integer and p ranges over all primes up to N. Though the proof for the Erdös-Straus conjecture is yet to be found, the existence of the upper and lower bounds shows that a typical prime has a small number of solutions to the Erdös-Straus equation.

As one of the earliest mathematical inventions, Egyptian fractions exhibit the contribution of ancient Egyptians to the dawn of mathematics. Aside from their historical value, Egyptian fractions also feature in many current conjectures. It is surprising to see that after so much progress in mathematics, we are still challenged by a concept that originated 4000 years ago.

REFERENCES

[1] Knott, Ron, *Egyptian Fractions*. University of Surrey, 2015.

[2] Tao, Terrence, and Elsholtz, Christian, "Counting the number of solutions to the Erdös-Straus Equation on unit fractions." *J. Austral. Math. Soc.* **94** (2013).

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