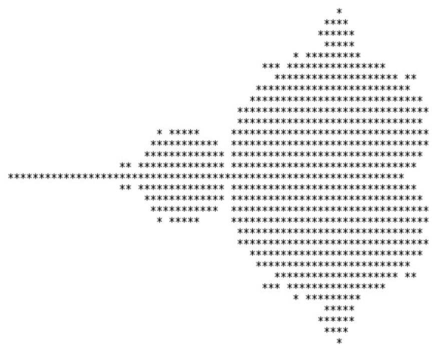

Figure 1. The first picture of the Mandelbrot set. Image from Elphaba via Wikipedia Commons.

1978: The Mandelbrot Set

While nowadays the Mandelbrot set is one of the most recognizable images in mathematics, the first image of it didn't appear until the work of Robert W. Brooks and Peter Matelski [**1**] in 1978; compare the resolution and detail available then (Figure 1) and now (Figure 2)!

The Mandelbrot set is an excellent example of a *fractal*, which is an object possessing a lot of self-similarities as we zoom in on various parts of it. It is constructed as follows. For each complex number c we form the sequence of points $\{z_{n;c}\}$, where $z_{0;c} = c$ and $z_{n+1;c} = z_{n;c}^2 + c$. The simplest pictures are obtained by coloring a point c black if the sequence is bounded and blue otherwise; for finer detail, we can color points whose sequences are unbounded differently based on how many iterations are needed to exceed a given amount. There are many online videos of incredible zoom-ins of this set; see for example [**5**].

The Fractal Geometry course webpage from Yale University, available online at [**2**], is an outstanding resource with detailed articles, illustrations of fractals, discussions of their roles and applications, and links to several TED lectures; the reader is strongly encouraged to visit and spend a few hours browsing.

Centennial Problem 1978

Proposed by Steven J. Miller, Williams College.

A strong contender for this year's topic was the video game *Space Invaders* [**6**]. Created by Tomohiro Nishikado and released in 1978, this mega-blockbuster game revolutionized the industry. Interestingly, though, one of the defining features of the game is due to hardware limitations of the time. Specifically, in the game alien ships are descending on the Earth. As more of them are destroyed, the remaining ships start traveling faster and faster until the last ship or two move at incredible speeds. This feature was due to difficulties in rendering the ships; the fewer ships there were that needed to be drawn, the faster the computer could display them! Nishikado decided that he liked this, and incorporated it into the game.

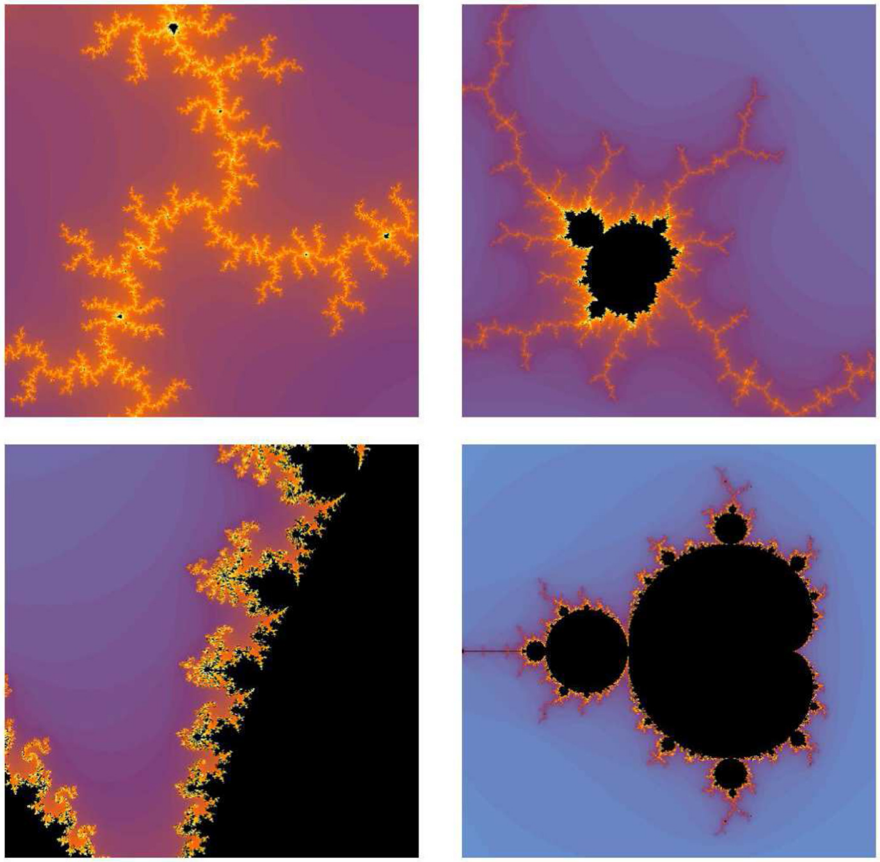

Figure 2. Several images of the Mandelbrot set.

The relation between Space Invaders and the Mandelbrot set is the need to render images quickly. With the reach of modern video gaming and computer animated movies, this is a multi-billion dollar a year issue. Instead of the Mandelbrot set one could look at other polynomial iterations instead of $f_c(z) = z^2 + c$. As we need to evaluate many polynomials time and time again, it becomes imperative to find as fast of a way of doing this as possible. If we have a polynomial $f(x) = a_n x^n + \cdots + a_0$, then brute force requires $n(n+1)/2$ multiplications and n additions. To see this, note that it costs us k multiplications to compute $a_k x^k$, and $1 + \cdots + n = n(n+1)/2$. It is possible to evaluate this polynomial in significantly fewer multiplications (multiplication is far more expensive than addition; compare how many digit operations are needed to multiply two n digit numbers versus adding them). By cleverly grouping terms, what is the fewest number of multiplications needed to evaluate $f(x)$? Amazingly, we can replace the order n^2, which we get by brute force evaluation, with a bound which is order n. (Spoiler alert: the answer is Horner's algorithm, which you may have seen years ago in grade school.) For high degree polynomials this is an incredible savings; for small degree it still adds up when we have to compute as often as we do in these detailed images. For a related problem, see the entry on the Fast Fourier Transform, and the discussion of the Strassen algorithm for fast matrix multiplication, from the 1965 entry.

Bibliography

[1] R. Brooks and P. Matelski, "The dynamics of 2-generator subgroups of PSL(2,C)", Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference, Irwin Kra and Bernard Maskit, editors, Princeton University Press, 1978.

[2] M. Frame, B. Mandelbrot and N. Neger, "Fractal Geometry", Yale University, accessed February 17, 2014 from http://classes.yale.edu/Fractals/. See in particular http://classes.yale.edu/Fractals/MandelSet/welcome.html.

[3] B. Mandelbrot, "The Fractal Geometry of Nature", W. H. Freeman, New York, 1982.

[4] Team Fresh, "Last Lights On - Mandelbrot fractal zoom to 6.066 e228 ($2\hat{7}60$)". http://vimeo.com/12185093.

[5] Wikipedia, "Mandelbrot set", http://en.wikipedia.org/wiki/Mandelbrot_set.

[6] Wikipedia, "Space invaders", http://en.wikipedia.org/wiki/Space_Invaders.